LAND USE PLANNING PROBLEM: A PRIMAL-DUAL SPLITTING ALGORITHM

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ABSTRACT

We propose a convex optimization urban planning problem for a wide class of objective functions. The dual of this problem is computed and the existence and uniqueness of the primal-dual solution are guaranteed under suitable conditions. A convergent algorithm is proposed, which solves the primal and dual problems simultaneously. Finally, our framework is illustrated by an application for the case where the planning goal is to attain spatial socio-economic homogeneity and numerical simulations are implemented.

Keywords: convex optimization, land use planning, proximity operators, splitting algorithms

1 INTRODUCTION

Megacities face chronic problems like congestion, segregation, urban sprawl, and high land rents, in addition to crime and the recent concern about climate change. These can be seen as costs of development, much of which occurs in cities, but they are also a complex challenge for the decision maker (city planner). Methods to study how to plan cities have so far concentrated on simulating the long term impacts of project and policies defined as future scenarios. Land use and transport models contribute in this task forecasting the impact of each scenario considered and is fair to say that these models have advanced in the last two decades to become operational and widely used by practitioners, which can be assessed from reviews in Wegener (1994, 1998), Hunt *et al* (2005), Timmermans and Zhang (2009) and Preston *et al* (2010).

However, the scenario approach leaves the planner with the enormous task of building wise scenarios. This is a complex task because potential subsidies and projects in the urban context can be a large number and testing each scenario is computationally costly. Consider for example the problem of a planner with some specific goal for the city armed with power of setting location subsidies and/or taxes to attain such goal. This policy yields a combinatorial number of subsidies/taxes considering options of agents and locations to be taxed; for a population size C the estimate for the number of subsidies/taxes rise to $N \times C$, where N is the number of locations. Consider now the goal of managing vehicles congestion in a network with L links and C vehicle types and M socio-economic groups, then the combinatorial number of link charges is of the order of $L \cdot C \cdot M$. In sum, the problem of setting policies/projects scenarios that contribute to optimize the city performance for some given goal and then test each scenario using land use and transport models is not really feasible; there is a need to develop models able to optimize the city by efficiently searching in the large domain of policies and projects.

The objective of this paper is to contribute in the development of methods able to optimize the city performance given the planners objective. In this general aim, our approach avoids defining a specific objective function, instead it defines a class of functions on which we can apply our method to optimize a city by proceeding in four steps. In the first step we use either the model developed in Briceño-Arias *et al* (2008), for the case without externalities, or the model in Bravo *et al* (2010) to include externalities, which solves the equilibrium problem yielding the location $\overline{\boldsymbol{x}}$ and travel times $\overline{\boldsymbol{t}}$ without any policy. The second step, which is the matter of this paper, aims at finding a location optimum $\boldsymbol{x}^*(\overline{\boldsymbol{t}})$, for a given set of travel times and a given objective function belonging to the class mentioned above and defined in the following section. We call this step the land use planning problem. Step three remains for future work, where we will extend our methodology to optimize the system integrating land use and transport to obtain $\boldsymbol{x}^*(\boldsymbol{t}^*)$, which is a reachable point of the real system, by means of introducing optimal policies. Finally, also in future work, we have to find the set of subsidies/taxes by location and agent (households and firms) that induces the land use and transport equilibrium to be the optimum $\boldsymbol{x}^*$.

The land use planning problem is not realistic because it considers travel times $\overline{\boldsymbol{t}}$ as given, and we know that they depend on the location pattern. But this problem is theoretically relevant as a step towards solving the integrated land use and transport optimal policies for a large set of objectives of the planner. This planning problem is formulated in Section in terms of its primal and dual problems and in Section we prove existence and uniqueness of the optimal primal-dual solution under suitable conditions. Next, in Section we propose a primal-dual splitting algorithm inspired on Briceño-Arias and Combettes (2011), which converges to the unique primal-dual solution. In Section , we examine an application for the case where the planning goal is to attain spatial socio-economic homogeneity and, finally, in Section , we provide numerical simulations revealing the improvements that our approach can contribute in terms of homogeneity. Let us start with some notation and preliminaries.

Notation. Let $(\mathcal{H}, \|\cdot\|)$ be a finite dimensional Euclidean space and denote by $\Gamma_0(\mathcal{H})$ the family of lower semicontinuous convex functions $\varphi\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ such that $\operatorname{dom}\varphi = \big\{x \in \mathcal{H} \;\big|\; \varphi(x) < +\infty\big\} \neq \varnothing$. A function $\varphi\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ is coercive if $\lim_{\|x\|\to+\infty}\varphi(x) = +\infty$. Now, let $\varphi \in \Gamma_0(\mathcal{H})$. The conjugate of φ is the function $\varphi^* \in \Gamma_0(\mathcal{H})$ defined by $\varphi^*\colon u \mapsto \sup_{x\in\mathcal{H}}(\langle x \mid u\rangle - \varphi(x))$. Moreover, for every $x \in \mathcal{H}$, $\varphi + \|x - \cdot\|^2/2$ possesses a unique minimizer, which is denoted by $\operatorname{prox}_\varphi x$. Alternatively,

$$\operatorname{prox}_\varphi = (\operatorname{Id} + \partial\varphi)^{-1}, \tag{1}$$

where $\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \;\big|\; (\forall y \in \mathcal{H}) \quad \langle y - x \mid u\rangle + \varphi(x) \leq \varphi(y)\big\}$ is the subdifferential of φ. In the particular case when φ is differentiable in some subset C of $\mathcal{H}$, we have, for every $x \in C$, $\partial\varphi(x) = \{\nabla\varphi(x)\}$. For every convex subset C of $\mathcal{H}$, the indicator function of C, denoted by ι_C, is the function which is 0 in C and $+\infty$ in $\mathcal{H} \setminus C$.

The following result will be useful in the following sections and some parts of it can be derived from the proof given in Rockafellar (1970, Corollary 26.3.1). For the sake of completeness we provide the proof.

Lemma 1 *Let* $\psi\colon \operatorname{dom}\psi \subset \mathbb{R} \to \left]-\infty,+\infty\right]$ *be strictly convex, differentiable in* $\operatorname{int}\operatorname{dom}\psi$*, and such that* $\operatorname{ran}(\psi') = \mathbb{R}$*. Then,* ψ^* *is strictly convex, differentiable in* $\operatorname{dom}\psi^* = \mathbb{R}$*, and* $\operatorname{ran}\psi^* \subset \operatorname{dom}\psi$*. Moreover,*

$$\psi^*\colon \eta \mapsto (\psi')^{-1}(\eta)\eta - \psi\big((\psi')^{-1}(\eta)\big) \quad \textit{and} \quad (\psi^*)' = (\psi')^{-1}. \tag{2}$$

Proof: Since ψ is strictly convex, differentiable in $\operatorname{int}\operatorname{dom}\psi$, and $\operatorname{ran}(\psi') = \mathbb{R}$, we have that $\psi'\colon \operatorname{int}\operatorname{dom}\psi \to \mathbb{R}$ is strictly increasing and surjective. Hence, $(\psi')^{-1}\colon \mathbb{R} \to \operatorname{int}\operatorname{dom}\psi$ exists, it is strictly increasing too, and $\operatorname{dom}(\psi')^{-1} = \mathbb{R}$. Since Bauschke and Combettes (2011, Proposition 16.13) asserts that $(\psi')^{-1} = (\psi^*)'$, we conclude that ψ^* is strictly convex, differentiable, $\operatorname{ran}\psi^* \subset \operatorname{dom}\psi$, and $\operatorname{dom}\psi^* = \mathbb{R}$. Finally, (2) follows from Hiriart-Urruty and Lemaréchal (1993, Proposition 6.2.1). □

2 PRIMAL-DUAL FORMULATION

Let C be the set of types of households and suppose that one firm controls the real estate supply. For every $i \in N$, let $S_i \in \left]0,+\infty\right[$ be the supply in the node i and, for every $h \in C$, let $H_h \in \left]0,+\infty\right[$ be the demand of the households type h in the land use market. For every $h \in C$ and $i \in N$, we denote by x_{hi} the number of households type h localized in i and we set $z_{hi} \in \mathbb{R}$ be the utility perceived by the household h on the location i. These utilities are assumed to be constant and known. Hence, transportation costs, accessibility, location externalities, and other features affecting these utilities are assumed to be exogenous. Additionally we assume the market clearing condition $T = \sum_{i\in N} S_i = \sum_{h\in C} H_h$, i.e., we suppose that the number of households demanding for a location coincides with the number of available houses.

The assumptions on utilities and the latter condition are very restrictive. Indeed, externalities on location exist as well as interaction between transportation and land use. A justification for not including these interactions is that this problem can be seen in the short term modelling. On the other hand, a more realistic scenario should include excess of supply or demand. These modifications could be explored as part of further research.

Let us denote as $\mathscr{C}$ the class of functions $\psi\colon \mathbb{R}\times[0,+\infty[\,\to\,]-\infty,+\infty]$ such that

$$(\forall z\in\mathbb{R})\quad \begin{cases}\psi(z,\cdot) \text{ is strictly convex and}\\ \lim_{x\to+\infty}\psi(z,x)=+\infty.\end{cases} \tag{3}$$

This class defines the set of objective functions that we consider for solving the land use planning problem defined as follows.

Problem 2 (Land use planning problem) For every $h\in C$ and $i\in I$, let $z_{hi}\in\mathbb{R}$, let $\psi_{hi}\in\mathscr{C}$ such that $\operatorname{dom}\psi_{hi}(z_{hi},\cdot)=[0,a_{hi}[$, for some $a_{hi}\in\,]0,+\infty]$, let

$$\Xi = \Big\{\boldsymbol{x}\in\mathbb{R}^{|C|\times|N|}\;\Big|\;(\forall i\in N)\quad \sum_{h\in C} x_{hi}=S_i \quad\text{and}\quad (\forall h\in C)\quad \sum_{i\in N} x_{hi}=H_h\Big\}, \tag{4}$$

and suppose that

$$\Xi\cap \bigtimes_{h\in C}\bigtimes_{i\in N}[0,a_{hi}[\,\neq\varnothing. \tag{5}$$

The problem is to

$$\underset{\boldsymbol{x}\in\Xi}{\text{minimize}}\quad \sum_{h\in C}\sum_{i\in N}\psi_{hi}(z_{hi},x_{hi}). \tag{6}$$

In Problem 2 the objective function belongs to the class $\mathscr{C}$, that is, it is strictly convex and coercive. Additionally we allow to functions $(\psi_{hi})_{h\in C,\,i\in N}$ to have as domain all the positive real numbers ($a_{hi}=+\infty$) or, if necessary, have a restricted domain ($a_{hi}<+\infty$). The latter case is justified in cases in which the domain of the function has an upper bound. This is the case, for example, of travel time costs on arcs, which goes to infinity as the flow reaches the capacity of the arc.

Example 3 Let $\mu\in\,]0,+\infty[$. In the particular case when, for every $h\in C$ and $i\in N$, $\psi_{hi}\colon(z,x)\mapsto -xz+x(\ln x-1)/\mu\in\mathscr{C}$, Problem 2 becomes

$$\underset{\boldsymbol{x}\in\Xi}{\text{minimize}}\quad \sum_{h\in C}\sum_{i\in N} -x_{hi}z_{hi}+\frac{1}{\mu}x_{hi}(\ln x_{hi}-1), \tag{7}$$

which is the bid-rent equilibrium problem presented in Briceño-Arias *et al* (2008), the well known entropy maximizing problem. Hence, this equilibrium problem can be seen as a particular case of our framework.

For computing the dual formulation of Problem 2, we need the following definitions and preliminaries. Define

$$\begin{cases} \boldsymbol{\Psi}\colon \mathbb{R}^{|C|\times|N|} \to \left]-\infty,+\infty\right] \\ \qquad \boldsymbol{x} \mapsto \begin{cases} \displaystyle\sum_{h\in C}\sum_{i\in N}\psi_{hi}(z_{hi},x_{hi}), & \text{if } \boldsymbol{x}\in \displaystyle\bigtimes_{h\in C}\bigtimes_{i\in N}\operatorname{dom}\psi_{hi}(z_{hi},\cdot) \\ +\infty, & \text{otherwise} \end{cases} \\ \boldsymbol{\Lambda}\colon \mathbb{R}^{|C|\times|N|} \to \mathbb{R}^{|C|+|N|} \\ \qquad \boldsymbol{x} \mapsto \left(\left(\displaystyle\sum_{i\in N}x_{hi}\right)_{h\in C}, \left(\displaystyle\sum_{h\in C}x_{hi}\right)_{i\in N}\right), \end{cases} \tag{8}$$

where $\boldsymbol{x}=(x_{hi})_{h\in C,\, i\in N}$ is a generic element of $\mathbb{R}^{|C|\times|N|}$.

Proposition 4 *Let $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ be defined as in* (8)*. Then, the following statements hold.*

(i) *$\boldsymbol{\Psi}$ is strictly convex, coercive, and*

$$(\forall\gamma\in\left]0,+\infty\right[)\quad \operatorname{prox}_{\gamma\boldsymbol{\Psi}} = \big(\operatorname{prox}_{\gamma\psi_{hi}(z_{hi},\cdot)}\big)_{h\in C,\, i\in N}. \tag{9}$$

(ii) *We have*

$$\boldsymbol{\Psi}^*\colon \mathbb{R}^{|C|\times|N|}\to\left]-\infty,+\infty\right] : \boldsymbol{u}\mapsto \sum_{h\in C}\sum_{i\in N}\varphi_{hi}(z_{hi},u_{hi}), \tag{10}$$

where, for every $h\in C$ and $i\in N$,

$$\varphi_{hi}\colon (z_{hi},u)\mapsto \psi_{hi}(z_{hi},\cdot)^*(u) = \sup_{x\in[0,a_{hi}[}\big(\langle x\mid u\rangle - \psi_{hi}(z_{hi},x)\big) \tag{11}$$

is differentiable. Moreover, $\boldsymbol{\Psi}^*$ is differentiable and $\nabla\boldsymbol{\Psi}^* = (\varphi_{hi}(z_{hi},\cdot)')_{h\in C,i\in N}$. In addition, suppose that, for every $h\in C$ and $i\in N$, $\psi_{hi}(z_{hi},\cdot)$ is differentiable in $]0,a_{hi}[$ and $\operatorname{ran}(\psi_{hi}(z_{hi},\cdot)')=\mathbb{R}$. Then, for every $h\in C$ and $i\in N$,

$$(\forall u\in\mathbb{R})\quad \varphi_{hi}(z_{hi},u) = (\psi_{hi}(z_{hi},\cdot)')^{-1}(u)u - \psi_{hi}\big(z_{hi},(\psi_{hi}(z_{hi},\cdot)')^{-1}(u)\big), \tag{12}$$

$\varphi_{hi}(z_{hi},\cdot)' = (\psi_{hi}(z_{hi},\cdot)')^{-1}$, and $\boldsymbol{\Psi}^*$ is strictly convex.

(iii) *$\boldsymbol{\Lambda}$ is linear, bounded, $\boldsymbol{\Lambda}^*\colon(\boldsymbol{b},\boldsymbol{r})\mapsto(b_h+r_i)_{h\in C,\, i\in N}$, where $(\boldsymbol{b},\boldsymbol{r})=((b_h)_{h\in C},(r_i)_{i\in N})$ is a generic element of $\mathbb{R}^{|C|+|N|}$, and $\|\boldsymbol{\Lambda}\|=\sqrt{|C|+|N|}$.*

Proof: (i): Is a consequence of (8), the properties of the class $\mathscr{C}$ in (3), and Combettes and Wajs(2005, Lemma 2.9). (ii): It follows from Bauschke and Combettes (2011, Proposition 13.27) that $\boldsymbol{\Psi}^*=\sum_{h\in C}\sum_{i\in N}\psi_{hi}(z_{hi},\cdot)^* = \sum_{h\in C}\sum_{i\in N}\varphi_{hi}(z_{hi},\cdot)$. The differentiability follows from Hiriart-Urruty and Lemaréchal (1993, Proposition 6.2.1) and the last result follows from Lemma 1. (iii): It is clear that $\boldsymbol{\Lambda}$ is linear and bounded. For every $\boldsymbol{x}\in\mathbb{R}^{|C|\times|N|}$ and $(\boldsymbol{b},\boldsymbol{r})\in\mathbb{R}^{|C|+|N|}$, we have

$$\langle(\boldsymbol{b},\boldsymbol{r})\mid\boldsymbol{\Lambda}\boldsymbol{x}\rangle = \sum_{h\in C}b_h\left(\sum_{i\in N}x_{hi}\right)+\sum_{i\in N}r_i\left(\sum_{i\in N}x_{hi}\right) = \sum_{h\in C}\sum_{i\in N}x_{hi}(b_h+r_i) = \langle\boldsymbol{\Lambda}^*(\boldsymbol{b},\boldsymbol{r})\mid\boldsymbol{x}\rangle. \tag{13}$$

On the other hand, using the inequality $2xy \leq x^2 + y^2$ we obtain, for every $\boldsymbol{x} \in \mathbb{R}^{|C|\times|N|}$,

$$\begin{aligned}
\|\boldsymbol{\Lambda x}\|^2 &= \sum_{h\in C}\left(\sum_{i\in N} x_{hi}\right)^2 + \sum_{i\in N}\left(\sum_{h\in C} x_{hi}\right)^2 \\
&= \sum_{h\in C}\left(\sum_{i\in N} x_{hi}^2 + \sum_{j\neq i} 2x_{hi}x_{hj}\right) + \sum_{i\in N}\left(\sum_{h\in C} x_{hi}^2 + \sum_{g\neq h} 2x_{hi}x_{gi}\right) \\
&\leq \sum_{h\in C}\left(\sum_{i\in N} x_{hi}^2 + (|N|-1)\sum_{i\in N} x_{hi}^2\right) + \sum_{i\in N}\left(\sum_{h\in C} x_{hi}^2 + (|C|-1)\sum_{h\in C} x_{hi}^2\right) \\
&= (|C|+|N|)\|\boldsymbol{x}\|^2, \qquad (14)
\end{aligned}$$

which yields $\|\boldsymbol{\Lambda}\| \leq \sqrt{|C|+|N|}$. The equality follows by taking, in particular, for every $(h,i) \in C \times N$, $x_{hi} = 1$, which yields

$$\|\boldsymbol{\Lambda x}\|^2 = \sum_{h\in C} |N|^2 + \sum_{i\in N} |C|^2 = (|C|+|N|)|C||N| = (|C|+|N|)\|\boldsymbol{x}\|^2. \qquad (15)$$

Hence $\|\boldsymbol{\Lambda}\| = \sqrt{|C|+|N|}$. □

Proposition 5 *Under the assumptions of Problem 2, the dual problem associated to* (6) *is*

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in\mathbb{R}^{|C|+|N|}}{\text{minimize}} \; \boldsymbol{\Phi}(\boldsymbol{b},\boldsymbol{r}) := \sum_{h\in C} H_h b_h + \sum_{i\in N} S_i r_i + \sum_{h\in C}\sum_{i\in N} \varphi_{hi}(z_{hi}, -b_h - r_i), \qquad (16)$$

where $(\varphi_{hi})_{h\in C,\, i\in N}$ *are defined in* (11)

Proof: Indeed, Problem 2 can be written equivalently as

$$\underset{\substack{\boldsymbol{x}\in\mathbb{R}^{|C|\times|N|}\\ \boldsymbol{\Lambda x}=(\boldsymbol{H},\boldsymbol{S})}}{\text{minimize}} \; \boldsymbol{\Psi}(\boldsymbol{x}), \qquad (17)$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ are defined in (6). Therefore, from Bauschke and Combettes (2011, Proposition 19.19) we have that the dual problem is

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in\mathbb{R}^{|C|+|N|}}{\text{minimize}} \; \boldsymbol{\Psi}^*(-\boldsymbol{\Lambda}^*(\boldsymbol{b},\boldsymbol{r})) + \langle(\boldsymbol{b},\boldsymbol{r}) \mid (\boldsymbol{H},\boldsymbol{S})\rangle, \qquad (18)$$

or equivalently, from Proposition 4(ii)–(iii),

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in\mathbb{R}^{|C|+|N|}}{\text{minimize}} \; \sum_{h\in C}\sum_{i\in N} \varphi_{hi}(z_{hi}, -b_h - r_i) + \sum_{h\in C} H_h b_h + \sum_{i\in N} S_i r_i, \qquad (19)$$

and the proof is finished. □

Remark 6 Note that, for every $(\boldsymbol{b},\boldsymbol{r}) \in \mathbb{R}^{|C|+|N|}$ and $\alpha \in \mathbb{R}$, $\boldsymbol{\Phi}(\boldsymbol{b}+\alpha, \boldsymbol{r}-\alpha) = \boldsymbol{\Phi}(\boldsymbol{b},\boldsymbol{r})$, where $b+\alpha = (b_1+\alpha, \ldots, b_{|C|}+\alpha)$ and $r-\alpha = (r_1-\alpha, \ldots, r_{|N|}-\alpha)$, which follows from the market clearing assumption $\sum_{h\in C} H_h = \sum_{i\in N} S_i$. Hence, for having uniqueness of the solution, we have to consider some additional constraints in the dual problem.

Remark 7 Let $\mu \in \,]0,+\infty[$. In the particular case when, for every $h \in C$ and $i \in N$, $\psi_{hi}\colon (z,x) \to -xz + x(\ln x - 1)/\mu \in \mathscr{C}$, (16) becomes

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in\mathbb{R}^{|C|+|N|}}{\text{minimize}} \quad \sum_{h\in C} H_h b_h + \sum_{i\in N} S_i r_i + \frac{1}{\mu}\sum_{h\in C}\sum_{i\in N} e^{\mu(z_{hi}-b_h-r_i)}, \tag{20}$$

which is the dual problem associated to the well known entropy maximization problem (7) (see Briceño-Arias *et al*, 2008). The first order optimality conditions of (20) are

$$(\forall h \in C)(\forall i \in N) \quad \begin{cases} \sum_{i\in N} e^{\mu(z_{hi}-b_h-r_i)} = H_h \\ \sum_{h\in C} e^{\mu(z_{hi}-b_h-r_i)} = S_i \end{cases} \tag{21}$$

and we deduce that the solution to the primal problem is $x_{hi} = e^{\mu(z_{hi}-b_h-r_i)}$.

3 EXISTENCE AND UNIQUENESS OF SOLUTIONS

Proposition 8 *Problem 2 has a unique solution* $(\overline{x}_{hi})_{h\in C,\, i\in N}$. *In addition, let* $\eta \in \mathbb{R}$ *and suppose that the dual problem considers one of the following constraints:*

(i) $\boldsymbol{b} \in D_1 = \big\{\boldsymbol{b} \in \mathbb{R}^{|C|} \;\big|\; b_1 = \eta\big\}$.

(ii) $\boldsymbol{b} \in D_2 = \big\{\boldsymbol{b} \in \mathbb{R}^{|C|} \;\big|\; \frac{1}{|C|}\sum_{h\in C} b_h = \eta\big\}$.

(iii) $\boldsymbol{r} \in D_3 = \big\{\boldsymbol{r} \in \mathbb{R}^{|N|} \;\big|\; r_1 = \eta\big\}$.

(iv) $\boldsymbol{r} \in D_4 = \big\{\boldsymbol{r} \in \mathbb{R}^{|N|} \;\big|\; \frac{1}{|N|}\sum_{i\in N} r_i = \eta\big\}$.

Then the dual problem (16) *has a unique solution* $((\overline{b}_h)_{h\in C}, (\overline{r}_i)_{i\in N})$. *Moreover, for every* $h \in C$ *and* $i \in N$, $\overline{x}_{hi} = \big(\varphi_{hi}(z_{hi},\cdot)\big)'(\overline{b}_h + \overline{r}_i)$, *where* $(\varphi_{hi}(z_{hi},\cdot))_{h\in C, i\in N}$ *are defined in* (11).

Proof: Since $\boldsymbol{\Psi} \in \Gamma_0(\mathbb{R}^{|C|\times|N|})$ is coercive, Ξ is closed and convex, and (5) yields $\Xi \cap \operatorname{dom} \boldsymbol{\Psi} \neq \varnothing$, Bauschke and Combettes (2011, Proposition 11.4(i)) asserts that the primal problem has solutions. It follows from (17) that Problem 2 can be written equivalently as

$$\underset{\boldsymbol{x}\in\mathbb{R}^{|C|\times|N|}}{\text{minimize}} \;\; \boldsymbol{\Psi}(\boldsymbol{x}) + \iota_{\{(\boldsymbol{H},\boldsymbol{S})\}}(\boldsymbol{\Lambda}\boldsymbol{x}). \tag{22}$$

Note that $\iota_{\{(\boldsymbol{H},\boldsymbol{S})\}} \in \Gamma_0(\mathbb{R}^{|C|+|N|})$ is polyhedral. Now, since $(\boldsymbol{H},\boldsymbol{S}) \in \,]0,+\infty[^{|C|+|N|}$, it follows from (5) and (8) that

$$(\boldsymbol{H},\boldsymbol{S}) \in \operatorname{int}\left(\boldsymbol{\Lambda}\left(\bigtimes_{h\in C}\bigtimes_{i\in N} [0, a_{hi}[\right)\right) = \bigtimes_{h\in C}]0, a_h[\times \bigtimes_{i\in N}]0, a_i[\,, \tag{23}$$

where, for every $h \in C$, $a_h = \sum_{i\in I} a_{hi}$ and, for every $i \in N$, $a_i = \sum_{h\in C} a_{hi}$. Hence, from Bauschke and Combettes (2011, Fact 15.25) we have $\inf(\boldsymbol{\Psi} + \iota_{\{(\boldsymbol{H},\boldsymbol{S})\}} \circ \boldsymbol{\Lambda}) = -\min(\boldsymbol{\Psi}^* \circ -\boldsymbol{\Lambda}^* +$

$\iota^*_{\{(\boldsymbol{H},\boldsymbol{S})\}}$). Therefore, we have existence of solutions to the dual problem. Moreover, it follows from Proposition 4(ii) that $\boldsymbol{\Psi}^*$ is differentiable in $\mathbb{R}^{|C|+|N|}$. Altogether, Bauschke and Combettes (2011, Proposition 19.3) asserts that Problem 2 has a unique solution

$$\overline{\boldsymbol{x}} = \nabla\boldsymbol{\Psi}^*(\boldsymbol{\Lambda}^*(\overline{\boldsymbol{b}},\overline{\boldsymbol{r}})), \tag{24}$$

where $(\overline{\boldsymbol{b}},\overline{\boldsymbol{r}})$ is a solution to the dual problem (16). Moreover, it follows from Lemma 1 and Proposition 4(iii) that (24) is equivalent to

$$(\forall h\in C)(\forall i\in N)\quad \overline{x}_{hi} = (\psi_{hi}(z_{hi},\cdot)^*)'(\overline{b}_h+\overline{r}_i) = \big(\varphi_{hi}(z_{hi},\cdot)\big)'(\overline{b}_h+\overline{r}_i). \tag{25}$$

Finally let us prove that, under one of the constraints (i)–(iv), $\boldsymbol{\Phi}$ is strictly convex and, hence, the dual problem (16) has a unique solution. Indeed, it follows from Lemma 1 that, for every $h\in C$ and $i\in N$, φ_{hi} is strictly convex. Let $(\boldsymbol{b}^1,\boldsymbol{r}^1)\neq(\boldsymbol{b}^2,\boldsymbol{r}^2)$ be vectors in $\mathbb{R}^{|C|+|N|}$ and let $\alpha\in\,]0,1[$. We have

$$\begin{aligned}
\boldsymbol{\Phi}(\alpha(\boldsymbol{b}^1,\boldsymbol{r}^1)+(1-\alpha)(\boldsymbol{b}^2,\boldsymbol{r}^2)) &= \sum_{h\in C} H_h(\alpha b_h^1+(1-\alpha)b_h^2) + \sum_{i\in N} S_i(\alpha r_i^1+(1-\alpha)r_i^2)\\
&\quad + \sum_{h\in C}\sum_{i\in N}\varphi_{hi}\big(z_{hi}, -(\alpha b_h^1+(1-\alpha)b_h^2)-(\alpha r_i^1+(1-\alpha)r_i^2)\big)\\
&= \alpha\Big(\sum_{h\in C} H_h b_h^1 + \sum_{i\in N} S_i r_i^1\Big) + (1-\alpha)\Big(\sum_{h\in C} H_h b_h^2 + \sum_{i\in N} S_i r_i^2\Big)\\
&\quad + \sum_{h\in C}\sum_{i\in N}\varphi_{hi}\big(z_{hi}, \alpha(-b_h^1-r_i^1)+(1-\alpha)(-b_h^2-r_i^2)\big).
\end{aligned} \tag{26}$$

Since, for every $h\in C$ and $i\in N$, $\varphi_{hi}(z_{hi},\cdot)$ is strictly convex, it is enough to prove that, under one of the constraints (i)–(iv), there exist $h_0\in C$ and $i_0\in N$ such that $-b^1_{h_0}-r^1_{i_0}\neq -b^2_{h_0}-r^2_{i_0}$, in which case from (26) we obtain that

$$\boldsymbol{\Phi}(\alpha(\boldsymbol{b}^1,\boldsymbol{r}^1)+(1-\alpha)(\boldsymbol{b}^2,\boldsymbol{r}^2)) < \alpha\boldsymbol{\Phi}(\boldsymbol{b}^1,\boldsymbol{r}^1)+(1-\alpha)\boldsymbol{\Phi}(\boldsymbol{b}^2,\boldsymbol{r}^2), \tag{27}$$

and the result follows. Let us proceed by contradiction. Suppose that

$$(\forall h\in C)(\forall i\in N)\quad -b_h^1-r_i^1 = -b_h^2-r_i^2. \tag{28}$$

If (i) holds, we have $b_1^1=b_1^2=\eta$ and we deduce from (28) in the particular case $h=1$ that, for every $i\in N$, $r_i^1=r_i^2$. Hence, it follows again from (28) that, for every $h\in C\setminus\{1\}$, $b_h^1=b_h^2$, which contradicts $(\boldsymbol{b}^1,\boldsymbol{r}^1)\neq(\boldsymbol{b}^2,\boldsymbol{r}^2)$. Now suppose that (ii) holds. Then we have $\sum_{h\in C} b_h^1=\sum_{h\in C} b_h^2=\eta$ and, by summing in h in (28), we deduce, for every $i\in N$, $r_i^1=r_i^2$. The contradiction is obtained in the same way as before. The cases (iii) and (iv) are analogous. □

Remark 9 We deduce from Proposition 8, Example 3, and Remark 7 that (7) and (20) have unique solutions depending on the utilities $\boldsymbol{z}=(z_{hi})_{h\in C,\, i\in N}$ under one of the additional conditions $\boldsymbol{b}\in D_1$, $\boldsymbol{b}\in D_2$, $\boldsymbol{r}\in D_3$, or $\boldsymbol{r}\in D_4$. We will denote by $(x_{k,hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$ and by $(b_{k,h}(\boldsymbol{z}))_{h\in C}$ and $(r_{k,i}(\boldsymbol{z}))_{i\in N}$ for $k\in\{1,\dots,4\}$ such solutions, respectively.

4 ALGORITHM AND CONVERGENCE

The algorithm proposed in this section for solving Problem 2 and its dual is a consequence of the following result which is derived from Briceño-Arias and Combettes (2011, Proposition 4.2). This method finds its roots in a splitting method for finding the zero of two maximally monotone operators provided in Tseng (2000), which is an extension of the well known proximal point method (Martinet, 1970 and Rockafellar, 1976).

Proposition 10 *Let $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ be two finite dimensional spaces, let $\Psi \in \Gamma_0(\boldsymbol{\mathcal{H}})$, let $\boldsymbol{u} \in \boldsymbol{\mathcal{G}}$, and let $\boldsymbol{\Lambda}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}$ be linear and bounded. Suppose that $\boldsymbol{\Lambda} \neq 0$ and that*

$$\operatorname{gra}(\partial\boldsymbol{\Psi}) \cap \big(\boldsymbol{\mathcal{H}} \times \operatorname{ran}(\boldsymbol{\Lambda}^*)\big) \neq \varnothing. \tag{29}$$

Consider the primal problem

$$\underset{\boldsymbol{x}\in\boldsymbol{\mathcal{H}}}{\operatorname{minimize}} \;\; \boldsymbol{\Psi}(\boldsymbol{x}) + \iota_{\{\boldsymbol{u}\}}(\boldsymbol{\Lambda x}), \tag{30}$$

and the dual problem

$$\underset{\boldsymbol{v}\in\boldsymbol{\mathcal{G}}}{\operatorname{minimize}} \;\; \boldsymbol{\Psi}^*(-\boldsymbol{\Lambda}^*\boldsymbol{v}) + \langle \boldsymbol{u} \mid \boldsymbol{v}\rangle. \tag{31}$$

Let $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ be an absolutely summable sequence in $\boldsymbol{\mathcal{H}}$, let $\boldsymbol{x}_0 \in \boldsymbol{\mathcal{H}}$, let $\boldsymbol{v}_0 \in \boldsymbol{\mathcal{G}}$, let $\varepsilon \in \left]0, 1/(\|\boldsymbol{\Lambda}\|+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\|\boldsymbol{\Lambda}\|\,]$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_{1,n} = \boldsymbol{x}_n - \gamma_n \boldsymbol{\Lambda}^*\boldsymbol{v}_n \\ \boldsymbol{p}_{1,n} = \operatorname{prox}_{\gamma_n \boldsymbol{\Psi}} \boldsymbol{y}_{1,n} + \boldsymbol{e}_n \\ \boldsymbol{p}_{2,n} = \boldsymbol{v}_n + \gamma_n(\boldsymbol{\Lambda x}_n - \boldsymbol{u}) \\ \boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \gamma_n(\boldsymbol{\Lambda p}_{1,n} - \boldsymbol{u}) \\ \boldsymbol{q}_n = \boldsymbol{p}_{1,n} - \gamma_n \boldsymbol{\Lambda}^*\boldsymbol{p}_{2,n} \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_{1,n} + \boldsymbol{q}_n. \end{array} \right. \tag{32}$$

Then the following statements hold for some solution $\overline{\boldsymbol{x}}$ to (30) *and some solution $\overline{\boldsymbol{v}}$ to* (31) *such that $-\nabla\boldsymbol{\Psi}(\overline{\boldsymbol{x}}) \in \operatorname{ran}\boldsymbol{\Lambda}^*$.*

(i) $\boldsymbol{x}_n - \boldsymbol{p}_{1,n} \to 0$ *and* $\boldsymbol{v}_n - \boldsymbol{p}_{2,n} \to 0$.

(ii) $\boldsymbol{x}_n \to \overline{\boldsymbol{x}}$, $\boldsymbol{p}_{1,n} \to \overline{\boldsymbol{x}}$, $\boldsymbol{v}_n \to \overline{\boldsymbol{v}}$, *and* $\boldsymbol{p}_{2,n} \to \overline{\boldsymbol{v}}$.

Proof: Since $\partial\iota_{\{\boldsymbol{u}\}} = \boldsymbol{\mathcal{H}}$, the results are direct consequence of Briceño-Arias and Combettes (2011, Proposition 4.2) when most of errors are zero. □

Proposition 11 *For every $h \in C$ and $i \in N$, let $(e_{hi,n})_{n\in\mathbb{N}}$ be an absolutely summable sequence in $\mathbb{R}$, let $x_{hi,0} \in \mathbb{R}$, let $(b_{h,0}, r_{i,0}) \in \mathbb{R}^2$, let $\varepsilon \in \left]0, 1/(\sqrt{|C|+|N|}+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence* *in* $[\varepsilon,(1-\varepsilon)/\sqrt{|C|+|N|}\,]$*, and set*

$$
(\forall n\in\mathbb{N})\quad
\left\lfloor
\begin{array}{l}
\text{For every } h\in C \text{ and } i\in N\\
\quad\left\lfloor
\begin{array}{l}
y_{1hi,n}=x_{hi,n}-\gamma_n(b_{h,n}+r_{i,n})\\
p_{1hi,n}=\operatorname{prox}_{\gamma_n\psi_{hi}(z_{hi},\cdot)}y_{1hi,n}+e_{hi,n}
\end{array}\right.\\
\text{For every } h\in C\\
\quad\left\lfloor
\begin{array}{l}
p_{2h,n}=b_{h,n}+\gamma_n\big(\sum_{i\in N}x_{hi,n}-H_h\big)\\
b_{h,n+1}=b_{h,n}+\gamma_n\big(\sum_{i\in N}p_{1hi,n}-H_h\big)
\end{array}\right.\\
\text{For every } i\in N\\
\quad\left\lfloor
\begin{array}{l}
p_{2i,n}=r_{i,n}+\gamma_n\big(\sum_{h\in C}x_{hi,n}-S_i\big)\\
r_{i,n+1}=r_{i,n}+\gamma_n\big(\sum_{h\in C}p_{1hi,n}-S_i\big)
\end{array}\right.\\
\text{For every } h\in C \text{ and } i\in N\\
\quad\left\lfloor
\begin{array}{l}
q_{hi,n}=p_{1hi,n}-\gamma_n(p_{2h,n}+p_{2i,n})\\
x_{hi,n+1}=x_{hi,n}-y_{1hi,n}+q_{hi,n}
\end{array}\right.
\end{array}\right.
\tag{33}
$$

Then the following statements hold for some solution $((\overline{x}_{hi})_{h\in C})_{i\in N}$ *to Problem 2 and some solution* $\big((\overline{b}_h)_{h\in C},(\overline{r}_i)_{i\in N}\big)$ *to its dual in* (16)*.*

(i) *For every* $h\in C$ *and* $i\in N$, $x_{hi,n}-p_{1hi,n}\to 0$, $b_{h,n}-p_{2h,n}\to 0$, *and* $r_{i,n}-p_{2i,n}\to 0$.

(ii) *For every* $h\in C$ *and* $i\in N$, $x_{hi,n}\to\overline{x}_{hi}$, $p_{1hi,n}\to\overline{x}_{hi}$, $b_{h,n}\to\overline{b}_h$, $p_{2h,n}\to\overline{b}_h$, $r_{i,n}\to\overline{r}_i$, *and* $p_{2i,n}\to\overline{r}_i$.

Proof: Let $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ as in (8). Condition (29) follows from Proposition 4(ii), and Proposition 4(iii) asserts that $\|\boldsymbol{\Lambda}\|=\sqrt{|C|+|N|}$. Moreover, we deduce from the proof of Proposition 5 that Problem 2 is equivalent to (30) and its dual is equivalent to (31). Therefore, the results are a consequence of (9) and Proposition 10 when $\boldsymbol{\mathcal{H}}=\mathbb{R}^{|C|\times|N|}$, $\boldsymbol{\mathcal{G}}=\mathbb{R}^{|C|+|N|}$, $\boldsymbol{u}=(\boldsymbol{H},\boldsymbol{S})$, for every $n\in\mathbb{N}$, $\boldsymbol{v}_n=(\boldsymbol{b}_n,\boldsymbol{r}_n)$, where $\boldsymbol{b}_n=(b_{h,n})_{h\in C}\in\mathbb{R}^{|C|}$ and $\boldsymbol{r}_n=(r_{i,n})_{i\in N}\in\mathbb{R}^{|N|}$. □

The difficulty of the algorithm proposed in Proposition 11 lies in the computation, for every $h\in C$, $i\in N$, and $n\in\mathbb{N}$, of $\operatorname{prox}_{\gamma_n\psi_{hi}(z_{hi},\cdot)}$. Several examples in which the proximity operator can be computed explicitly can be found in Combettes and Wajs (2005). The following result shows some interesting cases in which an explicit computation of the proximity operator can be obtained.

Lemma 12 *Let* $z\in\mathbb{R}$, $a\in\,]0,+\infty[$, $b\in\,]0,+\infty[$, $\gamma\in\,]0,+\infty[$, *and* $\mu\in\,]0,+\infty[$.

(i) *Let* $\psi\colon(z,x)\mapsto -zx+\frac{1}{\mu}x(\ln x-1)$. *Then* $\psi\in\mathscr{C}$ *and* $\operatorname{prox}_{\gamma\psi(z,\cdot)}\colon x\mapsto\frac{\gamma}{\mu}W\big(\frac{\mu}{\gamma}e^{\mu(x/\gamma+z)}\big)$, *where* W *is the product log function.*

(ii) *Let* $\psi\colon(z,x)\mapsto\iota_{[0,+\infty[}-zx+a(x-b)^2$. *Then* $\psi\in\mathscr{C}$ *and* $\operatorname{prox}_{\gamma\psi(z,\cdot)}\colon x\mapsto\max\{(x+\gamma z+2\gamma ab)/(1+2\gamma a),0\}$.

Proof: Let $(x,p)\in\mathbb{R}^2$. It is clear from (3) that both functions are in $\mathscr{C}$. (i): We have $p=\operatorname{prox}_{\gamma\psi(z,\cdot)}x\Leftrightarrow x-p=\gamma\psi(z,\cdot)'(p)\Leftrightarrow x+\gamma z=p+\frac{\gamma}{\mu}\ln p\Leftrightarrow\frac{\mu}{\gamma}x+\mu z=\frac{\mu}{\gamma}p+\ln p\Leftrightarrow e^{\mu(x/\gamma+z)}=pe^{\frac{\mu}{\gamma}p}\Leftrightarrow\frac{\mu}{\gamma}p=W\big(\frac{\mu}{\gamma}e^{\mu(x/\gamma+z)}\big)$, and the result follows. (ii): We have $p=\operatorname{prox}_{\gamma\psi(z,\cdot)}x\Leftrightarrow x-p\in$

$\gamma\partial\psi(z,\cdot)(p) \Leftrightarrow x-p \in N_{[0,+\infty[}(p)-\gamma z+2\gamma a(p-b) \Leftrightarrow x+\gamma z+2\gamma ab \in N_{[0,+\infty[}(p)+p(1+2\gamma a) \Leftrightarrow (x+\gamma z+2\gamma ab)/(1+2\gamma a) \in N_{[0,+\infty[}(p)+p \quad \Leftrightarrow \quad p = P_{[0,+\infty[}((x+\gamma z+2\gamma ab)/(1+2\gamma a))$, which yields the result. □

5 APPLICATION

We consider the case where the policy maker seeks the combined objective of maximizing agents utilities and minimizing a measure of spatial socio-economic homogeneity.

Consider the definitions and notations introduced in Section . A standard measure for the socio-economic homogeneity for a location $\boldsymbol{x} = (x_{hi})_{h\in C,\, i\in N} \in \Xi$ is

$$SI(\boldsymbol{x}) = \sum_{i\in N}\left(\frac{\sum_{h\in C} x_{hi} I_h}{S_i} - \overline{I}\right)^2, \tag{34}$$

where $\overline{I} = \sum_{h\in C} H_h I_h/T$ and, for every $h \in C$, $I_h \in \,]0,+\infty[$ is the average income of households type h. Instead of using this measure, in the following proposition we provide a related measure which is separable as the objective function in (6).

Proposition 13 *Let $\boldsymbol{x} = (x_{hi})_{h\in C,\, i\in N} \in \Xi$ and define the zone segregation level and the aggregated segregation level by*

$$(\forall i \in N) \quad SL_i(\boldsymbol{x}) = \sum_{h\in C} I_h (x_{hi}/S_i - H_h/T)^2 \quad \textit{and} \quad SL(\boldsymbol{x}) = \sum_{i\in N} SL_i(\boldsymbol{x}), \tag{35}$$

respectively. Then, $0 \le SI(\boldsymbol{x}) \le (\sum_{h\in C} I_h) SL(\boldsymbol{x})$ and the unique minimizer of SL, $\boldsymbol{x}_{SL} = \big((S_i H_h)/T\big)_{h\in C,\, i\in N}$, is a minimizer of SI.

Proof: Since the SL is strictly convex and coercive it has a unique minimizer $\boldsymbol{x}_{SL} \in \Xi$. Let $\boldsymbol{x}^* = \big((S_i H_h)/T\big)_{h\in C,\, i\in N}$. Since $SL(\boldsymbol{x}^*) = 0$ and SL is a positive function, it is clear that $\boldsymbol{x}_{SL} = \boldsymbol{x}^*$. Moreover, let $\boldsymbol{x} = (x_{hi})_{h\in C,\, i\in N} \in \Xi$. It follows from (34), (35), and Bauschke and Combettes (2011, Lemma 2.13(ii)) that

$$0 \le SI(\boldsymbol{x}) = \sum_{i\in N}\left(\sum_{h\in C} I_h (x_{hi}/S_i - H_h/T)\right)^2 \le \left(\sum_{h\in C} I_h\right) SL(\boldsymbol{x}) \tag{36}$$

and, hence, $SI(\boldsymbol{x}_{SL}) = 0$, which yields the result. □

The problem under consideration in this section is to find a location which minimizes the aggregated segregation level and, simultaneously, maximizes the total utility. More precisely,

$$\underset{\boldsymbol{x}\in\Xi\cap\mathbb{R}_+^{|C|\times|N|}}{\text{minimize}} \quad -\sum_{h\in C}\sum_{i\in N} x_{hi} z_{hi} + \frac{1}{\alpha}\sum_{h\in C}\sum_{i\in N} I_h (x_{hi}/S_i - H_h/T)^2, \tag{37}$$

where Ξ is defined in (4) (market clearing) and $\alpha > 0$. This parameter is a measure of the importance of the utility of households in the planning objective function. The higher is α, the higher is the importance of the utility for the planner. The lower is α, the higher is the importance of the segregation level for the planner.

Problem (37) is a particular case of Problem 2 when, for every $(h,i) \in C \times N$, $\psi_{hi}(z_{hi},\cdot)\colon x \mapsto -xz_{hi} + I_h(x/S_i - H_h/T)^2/\alpha$. It follows from Lemma 12(ii) that functions $(\psi_{hi})_{h\in C, i\in N}$ are in $\mathscr{C}$. Therefore, it follows from Proposition 8 that (37) has a unique primal solution $\boldsymbol{x}^*_\alpha(\boldsymbol{z})$, which is called *social optimum*, and a unique dual solution $(\boldsymbol{c}^*_\alpha(\boldsymbol{z}), \boldsymbol{d}^*_\alpha(\boldsymbol{z}))$ under the additional condition $d_{1,\alpha} = 0$. The following proposition asserts that, for α small enough, the dual solution does not depend on α. It also provides a connection between the aggregated segregation level in the social optimum solution to (37) and the parameter α.

Proposition 14 *Set $\alpha > 0$ small enough. Then, the dual solution does not depend on α, i.e., $(\boldsymbol{c}^*_\alpha(\boldsymbol{z}), \boldsymbol{d}^*_\alpha(\boldsymbol{z})) = (\boldsymbol{c}^*(\boldsymbol{z}), \boldsymbol{d}^*(\boldsymbol{z}))$. Moreover, the segregation level defined in* (35) *in the social optimum is*

$$SL(\boldsymbol{x}^*_\alpha(\boldsymbol{z})) = \frac{\alpha^2}{4}\sum_{h\in C}\sum_{i\in N}\frac{S_i^2}{I_h}(z_{hi} - c_h^*(\boldsymbol{z}) - d_i^*(\boldsymbol{z})). \tag{38}$$

Proof: Suppose that $\boldsymbol{x}^*_\alpha(\boldsymbol{z})$ is strictly feasible, i.e., that, for every $h \in C$ and $i \in N$, $x^*_{hi,\alpha}(\boldsymbol{z}) > 0$. Then, the first order conditions of (37) yield

$$(\forall h \in C)(\forall i \in N) \quad x^*_{hi,\alpha} = \frac{H_h S_i}{T} + \alpha\frac{S_i^2}{2I_h}(z_{hi} - c^*_{h,\alpha}(\boldsymbol{z}) - d^*_{i,\alpha}(\boldsymbol{z})), \tag{39}$$

where $(c^*_{h,\alpha}(\boldsymbol{z}))_{h\in C}$ and $(d^*_{i,\alpha}(\boldsymbol{z}))_{i\in N}$ are the Lagrange multipliers of the constraints in Ξ (dual solution). Imposing these constraints on the primal solution $(x^*_{hi,\alpha})_{h\in C,\, i\in N}$ we obtain

$$(\forall i \in N) \quad S_i = \sum_{h\in C} x^*_{hi,\alpha} = S_i + \alpha\frac{S_i^2}{2}\sum_{h\in C}\frac{(z_{hi} - c^*_{h,\alpha}(\boldsymbol{z}) - d^*_{i,\alpha}(\boldsymbol{z}))}{I_h} \tag{40}$$

$$(\forall h \in C) \quad H_h = \sum_{i\in N} x^*_{hi,\alpha} = H_h + \alpha\frac{\alpha}{2I_h}\sum_{i\in N} S_i^2(z_{hi} - c^*_{h,\alpha}(\boldsymbol{z}) - d^*_{i,\alpha}(\boldsymbol{z})), \tag{41}$$

which yields that, under the additional condition $d_1 = 0$, the dual solution is the unique solution to the system

$$(\forall h \in C) \quad c_h = \sum_{i\in N}(z_{hi} - d_i)\delta_i \tag{42}$$

$$(\forall i \in N) \quad d_i = \sum_{h\in C}(z_{hi} - c_h)\beta_h, \tag{43}$$

where, for every $h \in C$ and $i \in N$, $\beta_h = I_h^{-1}/\sum_{g\in C} I_g^{-1}$ and $\delta_i = S_i^2/\sum_{j\in N} S_j^2$. Hence, the dual solution does not depend on α and it follows from (39) that the primal solution is strictly feasible for α small enough. Finally, (38) follows from a straightforward computation. □

Note that, the segregation level is increasing in α, which is natural from the definition of this parameter.

In addition, observe that (5) is easily satisfied since $\operatorname{dom}\psi_{hi}(z_{hi},\cdot) = [0,+\infty[$. Therefore, we can solve problem (37) by using the algorithm proposed in (33), which, by applying Lemma 12(ii), becomes (we set $e_{hi,n} \equiv 0$)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \text{For every } h \in C \text{ and } i \in N \\ \left\lfloor \begin{array}{l} y_{1hi,n} = x_{hi,n} - \gamma_n(b_{h,n} + r_{i,n}) \\ p_{1hi,n} = \max\{\big(y_{1hi,n} + \gamma_n z_{hi} + 2\gamma_n I_h H_h/(\alpha S_i T)\big)/\big(1 + 2\gamma_n I_h/(\alpha S_i^2)\big), 0\} \end{array} \right. \\ \text{For every } h \in C \\ \left\lfloor \begin{array}{l} p_{2h,n} = b_{h,n} + \gamma_n\big(\sum_{i\in N} x_{hi,n} - H_h\big) \\ b_{h,n+1} = b_{h,n} + \gamma_n\big(\sum_{i\in N} p_{1hi,n} - H_h\big) \end{array} \right. \\ \text{For every } i \in N \\ \left\lfloor \begin{array}{l} p_{2i,n} = r_{i,n} + \gamma_n\big(\sum_{h\in C} x_{hi,n} - S_i\big) \\ r_{i,n+1} = r_{i,n} + \gamma_n\big(\sum_{h\in C} p_{1hi,n} - S_i\big) \end{array} \right. \\ \text{For every } h \in C \text{ and } i \in N \\ \left\lfloor \begin{array}{l} q_{1hi,n} = p_{1hi,n} - \gamma_n(p_{2h,n} + p_{2i,n}) \\ x_{hi,n+1} = x_{hi,n} - y_{1hi,n} + q_{1hi,n}. \end{array} \right. \end{array} \right. \tag{44}$$

If the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is in $\big]0,(|C|+|N|)^{-1/2}\big[$, Proposition 11 asserts that, for every $h \in C$ and $i \in N$, the sequence $(x_{hi,n})_{n\in\mathbb{N}}$ converges to some x^*_{hi} and $\boldsymbol{x}^* = (x^*_{hi})_{h\in C,\, i\in N}$ is the solution to (37) and, additionally, the sequence $(b_{h,n}, r_{i,n})_{n\in\mathbb{N}}$ converges to some (b^*_h, r^*_i) and $((b^*_h)_{h\in C}, (r^*_i)_{i\in N})$ is a solution to the associated dual problem.

Remark 15 Note that in this case the proximal step (where $\operatorname{prox}_{\psi_{hi}(z_{hi},\cdot)}$ is computed) in the algorithm (33) is easy to compute numerically avoiding the use of any error term.

6 SIMULATIONS

In this section we provide simulations in a fictitious city in which the segregation is high. The purpose of these simulations is to show that the planning problem obtains a location which has better levels of homogeneity verifying the convergence of the algorithm, and to verify the dependence of the social homogeneity with respect to the parameter α. Hence, as an example, we take arbitrary values on the supply, demand (satisfying market clearing condition), and utility.

We model the location of the households by problem (7) and we compute its solution via the biproportional algorithm proposed in Macgill (1977). On the other hand, we solve the planning problem presented in (37) by the method in (44) for obtaining a location with less segregation. Afterwards we compare these solutions.

We consider a city with 10 zones ($|N| = 10$) and 5 types of households ($|C| = 5$). The convergence criteria of the algorithm for solving problem (7) is $\|\boldsymbol{b}_n - \boldsymbol{b}_{n+1}\|/\|\boldsymbol{b}_n\| \leq 10^{-10}$ and $\|\boldsymbol{r}_n - \boldsymbol{r}_{n+1}\|/\|\boldsymbol{r}_n\| \leq 10^{-10}$. The real estate supply per zone and the number of households per type are $S = (S_1,\ldots,S_{10}) = (25, 37, 24, 21, 34, 43, 23, 27, 20, 14)$ and $H = (H_1,\ldots,H_5) = (50, 56, 51, 60, 51)$, respectively, and the total supply (or demand) is $T = 268$. The average income of households per type is $I = (I_1,\ldots,I_5) = (2, 4, 6, 8, 10)$. Additionally, the utilities

Table 1: Utilities perceived by a household of every type for every zone.

h \ i	1	2	3	4	5	6	7	8	9	10
1	50	50	50	0	0	0	0	-50	-50	-50
2	50	50	0	0	0	0	0	-50	-50	-50
3	-50	-50	0	0	50	50	50	50	0	0
4	0	0	0	50	50	50	0	0	0	0
5	-50	-50	-50	0	0	0	50	50	50	50

$\boldsymbol{z} = (z_{hi})_{h\in C,\, i\in N}$ are presented in Table 1. We recall that, for every $h \in C$ and $i \in N$, z_{hi} represents the utility perceived by a household type h for a location in i.

Table 2 presents the equilibrium $\overline{\boldsymbol{x}}(\boldsymbol{z}) = (\overline{x}_{hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$ obtained by solving (7) via the algorithm in Macgill (1977) with $\mu = 5 \times 10^{-2}$. It also shows the segregation level of the equilibrium by zone and in the whole city computed by (35). Additionally, in Figure 1 we show the percentage of households of each type $h \in \{1, \ldots, 5\}$ located in every zone $i \in \{1, \ldots, 10\}$ in this case. We remark the very high segregation in all zones.

Table 2: Equilibrium $\overline{\boldsymbol{x}}(\boldsymbol{z})$ and segregation level.

h \ i	1	2	3	4	5	6	7	8	9	10	$SL(\overline{\boldsymbol{x}})$
1	9	13	19	2	2	3	2	0	0	0	
2	16	23	3	3	3	4	3	0	0	0	
3	0	0	1	1	11	14	9	13	1	1	
4	0	1	1	15	16	21	1	2	2	1	
5	0	0	0	1	1	1	9	12	16	11	
$SL_i(\overline{\boldsymbol{x}})$	1.64	1.64	1.57	2.17	1.00	1.00	0.89	1.61	4.13	4.13	**19.78**

In order to obtain a less segregated city, we consider the computation of the aggregated segregation level in terms of α obtained by (38) in Proposition 14. The quadratic dependence of the aggregated segregation level with respect to α is shown in Figure 3. Hence, for obtaining lower segregation values we need to use lower values of α in the method (44).

We deduce from Figure 3 that an aggregated segregation level lower to 20 (approximately the segregation of the equilibrium $\overline{\boldsymbol{x}}(\boldsymbol{z})$) is obtained by considering a value of α lower than 1.3×10^{-4}. Considering the much lower value of $\alpha = 3 \times 10^{-5}$, the solution $\boldsymbol{x}^*(\boldsymbol{z}) = (x^*_{hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$ to (37) obtained by the algorithm (44) and the corresponding segregation level in every zone and aggregated are presented in Table 3. In Figure 2 we exhibit the percentage of households of each type $h \in \{1, \ldots, 5\}$ located in every zone $i \in \{1, \ldots, 10\}$ in this case. We observe that the segregation level of $\boldsymbol{x}^*(\boldsymbol{z})$ is is drastically reduced in every zone and the aggregated segregation level reduces from 19.78 to 0.99. This coincides with the theoretical computation for $\alpha = 3 \times 10^{-5}$ provided in Proposition 14, as we can observe in Figure 3.

7 CONCLUSIONS

To the best of our knowledge, the land use planning problem defined in a discrete domain of locations and households remains open, so planners have no method to identify the optimum allocation of households for specific objectives.

This paper formulates the land use planning problem that faces a planner policies who seeks specifics objectives for the city. For a wide set of functions that represent different objectives, we have proved that a unique location solution exists, i.e. a distribution of households in space considering their socioeconomics differences, and we proposed an algorithm that converges to the solution of the planning problem. We test the algorithm for an hypothetical city and show that results can significantly improve the cities performance given the planner objective. The main limitations of our method is the assumption that travel decisions and transport costs are exogenous and there is now a need of a method to calculate optimal subsidies able to drive the land use market to an optimum; both are topics for future research.

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Tables and Figures

Table 3: Solution $\boldsymbol{x}^*$ and segregation level.

$h \backslash i$	1	2	3	4	5	6	7	8	9	10	$SL(\boldsymbol{x}^*)$
1	7	11	6	4	5	6	4	3	3	2	
2	7	13	5	4	6	7	4	4	3	2	
3	3	2	4	4	8	11	5	7	4	3	
4	5	7	5	5	9	12	4	5	4	3	
5	3	4	3	4	6	7	6	8	6	4	
$SL_i(\boldsymbol{x}^*)$	0.13	0.29	0.05	0.01	0.04	0.06	0.07	0.20	0.10	0.05	**0.99**

Figure 1: Percentage of types of households by zone for $\overline{\boldsymbol{x}}(\boldsymbol{z})$.

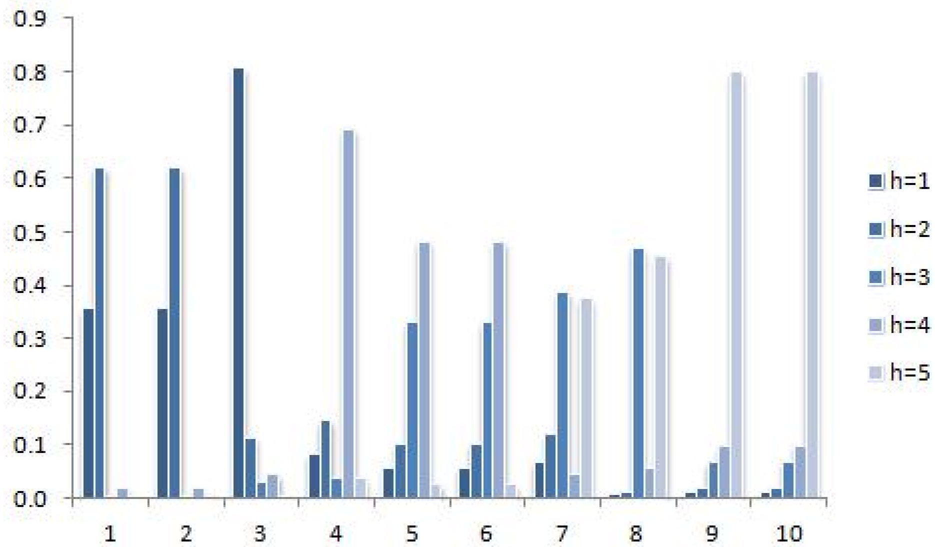

Figure 2: Percentage of types of households by zone for $\boldsymbol{x}^*(\boldsymbol{z})$.

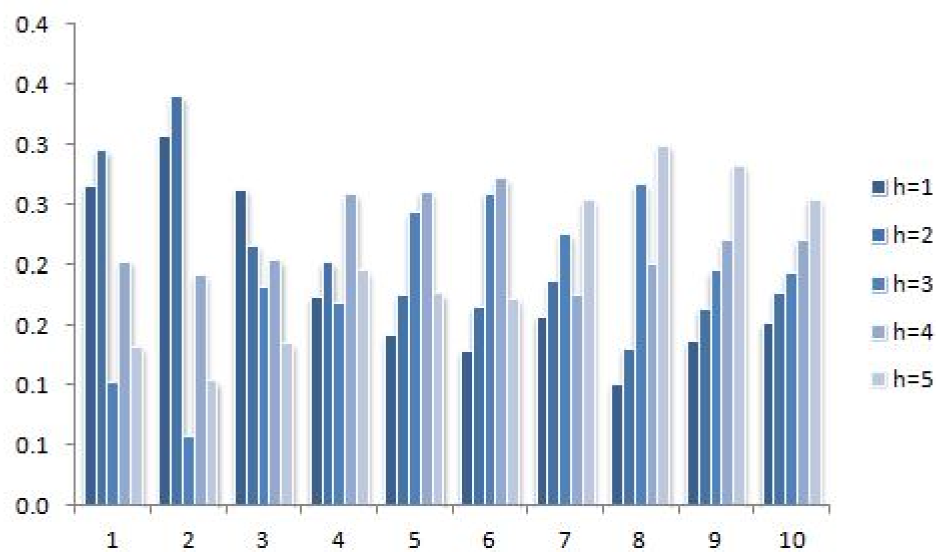

Figure 3: Function $\alpha(SL)$.